

# The Lichnerowicz-Obata theorem on sub-Riemannian manifolds with transverse symmetries

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## Abstract

We prove a lower bound for the first eigenvalue of the sub-Laplacian on sub-Riemannian manifolds with transverse symmetries. When the manifold is of  $H$ -type, we obtain a corresponding rigidity result: If the optimal lower bound for the first eigenvalue is reached, then the manifold is equivalent to a 1 or a 3-Sasakian sphere.

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## 1 Introduction

The study of optimal lower bounds for sub-Laplacians on manifolds has attracted a lot of interest in the past few years. In particular, the most studied example has been the example of the sub-Laplacian on CR manifolds. In that case, the story goes back at least to the work by Greenleaf [12] which has seen, since then, several improvements and variations. We mention in particular the works by Aribi-Dragomir-El Soufi [1], Barletta [2], Baudoin-Wang [7], Ivanov-Petkov-Vassilev [18, 19], Li [21] and Li-Luk [22]. Some optimal lower bounds for the first eigenvalue of sub-Laplacians also have been obtained in the context of quaternionic contact manifolds by Ivanov-Petkov-Vassilev [15, 16, 17]. More general situations were even considered by Hladky [13].

In the present work, we obtain optimal first eigenvalue lower bounds in a large class of sub-Riemannian manifolds that encompasses as a very special case Sasakian manifolds and 3-Sasakian manifolds. This class is the class of sub-Riemannian manifolds with transverse symmetries that was introduced in [4]. Roughly speaking, a sub-Riemannian manifold with transverse symmetries is a sub-Riemannian manifold for which the horizontal distribution admits a canonical intrinsic complement which is generated by sub-Riemannian Killing fields. The lower bound we obtain in that case improves a previous lower bound that was obtained by Baudoin-Kim in [5]. The method of [5] was to apply to an eigenfunction of the sub-Laplacian the curvature-dimension inequality proved in [4], and then to integrate this curvature-dimension inequality over the manifold. When used on a Riemannian manifold, this technique provides the optimal Lichnerowicz estimate. However, interestingly, this technique does not give the optimal estimate in the sub-Riemannian case and more work is needed. Our approach here, is to take advantage of the Bochner-Weitzenböck formula that was recently proved in [3] and to integrate this equality over the manifold. This gives an equality which when applied to an eigenfunction gives a better estimate than [5] for the first eigenvalue. In the 1 or the 3-Sasakian case, the lower bound we obtain coincides with the known optimal lower bound.

In the second part of the paper, we check the optimality of our lower bound, by proving a rigidity result in the spirit of Obata [23]. More precisely we prove the following result:

**Theorem 1.1** *Let  $\mathbb{M}$  be a compact sub-Riemannian manifold of  $H$ -type with dimension  $d + \mathfrak{h}$ ,  $d$  being the dimension of the horizontal bundle and  $\mathfrak{h}$  the dimension of the vertical bundle. Assume that for every smooth horizontal one-form  $\eta$ ,*

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta \rangle_{\mathcal{H}^*} \geq \rho \|\eta\|_{\mathcal{H}^*}^2,$$

*with  $\rho > 0$ , then the first eigenvalue  $\lambda_1$  of the sub-Laplacian  $-L$  satisfies*

$$\lambda_1 \geq \frac{\rho d}{d - 1 + 3\mathfrak{h}}.$$

*Moreover, if  $\lambda_1 = \frac{\rho d}{d - 1 + 3\mathfrak{h}}$ , then  $\mathbb{M}$  is equivalent to a 1-Sasakian sphere  $\mathbb{S}^{2m+1}(r)$  or a 3-Sasakian sphere  $\mathbb{S}^{4m+3}(r)$  for some  $r > 0$  and  $m \geq 1$ .*

This result for  $H$ -type manifolds generalizes the corresponding theorem for Sasakian manifolds by Chang-Chiu [9] and for 3-Sasakian manifolds by Ivanov-Petkov-Vassilev [16]. Like in the cited references, the main idea is to prove that an extremal eigenfunction  $f$  for the sub-Laplacian needs to satisfy  $\tilde{\nabla}^2 f = -\alpha f$ , for the Levi-Civita connection of a well chosen Riemannian extension of the sub-Riemannian metric. We can observe that in the works [18, 21] or [17] the Sasakian condition is not needed, it is therefore an interesting question to try to generalize our result to more general sub-Riemannian structures where the transverse symmetries condition is not assumed.

The paper is organized as follows. Section 2 presents the basic materials on sub-Riemannian manifolds with transverse symmetries. In particular, we present the Bochner-Weitzenböck formula that was proved in [3]. Section 3 is devoted to the proof of the lower bound for the first eigenvalue and Section 4 proves its optimality in the context of  $H$ -type manifolds.

## 2 The Bochner-Weitzenböck formula on sub-Riemannian manifolds with transverse symmetries

The notion of sub-Riemannian manifold with transverse symmetries was introduced in [4]. We recall here the main geometric quantities and operators related to this structure and we refer to [3] and [4] for further details. We in particular focus on the Bochner-Weitzenböck formula that was proved in [3].

Let  $\mathbb{M}$  be a smooth, connected manifold with dimension  $d + \mathfrak{h}$ . We assume that  $\mathbb{M}$  is equipped with a bracket generating distribution  $\mathcal{H}$  of dimension  $d$  and a fiberwise inner product  $g_{\mathcal{H}}$  on that distribution. The distribution  $\mathcal{H}$  is referred to as the set of *horizontal directions*, while a vector field which is tangent to  $\mathcal{H}$  is said to be horizontal.

**Definition 2.1** *It is said that  $\mathbb{M}$  is a sub-Riemannian manifold with transverse symmetries if there exists an  $\mathfrak{h}$ -dimensional Lie algebra  $\mathcal{V}$  of sub-Riemannian Killing vector fields such that for every  $x \in \mathbb{M}$ ,*

$$T_x\mathbb{M} = \mathcal{H}(x) \oplus \mathcal{V}(x).$$

We recall that a vector field  $Z$  is said to be a sub-Riemannian Killing vector field if the flow it generates locally preserves the horizontal distribution and induces a  $g_{\mathcal{H}}$ -isometry. Also  $\mathcal{V}$  denotes the distribution referred to as the set of *vertical directions*. The choice of an inner product  $g_{\mathcal{V}}$  on the Lie algebra  $\mathcal{V}$  naturally endows  $\mathbb{M}$  with a one-parameter family of Riemannian metrics that makes the decomposition  $\mathcal{H} \oplus \mathcal{V}$  orthogonal:

$$g_{\varepsilon} = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon} g_{\mathcal{V}}, \quad \varepsilon > 0.$$

For notational convenience, we will often use the notation  $\langle \cdot, \cdot \rangle_{\varepsilon}$ , resp.  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ , resp.  $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ , instead of  $g_{\varepsilon}$ , resp.  $g_{\mathcal{H}}$ , resp.  $g_{\mathcal{V}}$ . We can extend  $g_{\mathcal{H}}$  on  $T_x\mathbb{M} \times T_x\mathbb{M}$  by the requirement that  $g_{\mathcal{H}}(u, v) = 0$  whenever  $u$  or  $v$  is in  $\mathcal{V}(x)$ . We similarly extend  $g_{\mathcal{V}}$ . Hence for any  $u \in T_x\mathbb{M}$ ,

$$\|u\|_{\varepsilon}^2 = \|u\|_{\mathcal{H}}^2 + \frac{1}{\varepsilon} \|u\|_{\mathcal{V}}^2.$$

The volume measure obtained as a product of the horizontal volume measure determined by  $g_{\mathcal{H}}$  and the volume measure determined by  $g_{\mathcal{V}}$  will be denoted by  $\mu$  and is our reference measure on  $\mathbb{M}$ .

The following connection was introduced in [4].

**Proposition 2.2 (See [4])** *There exists a unique connection  $\nabla$  on  $\mathbb{M}$  satisfying the following properties:*

- (i)  $\nabla g_\varepsilon = 0$ ,  $\varepsilon > 0$ ;
- (ii) If  $X$  and  $Y$  are horizontal vector fields,  $\nabla_X Y$  is horizontal;
- (iii) If  $Z \in \mathcal{V}$ ,  $\nabla Z = 0$ ;
- (iv) If  $X, Y$  are horizontal vector fields and  $Z \in \mathcal{V}$ , the torsion vector field  $T(X, Y)$  is vertical and  $T(X, Z) = 0$ .

Intuitively  $\nabla$  is the connection which coincides with the Levi-Civita connection of the Riemannian metric  $g_\varepsilon$  on the horizontal bundle  $\mathcal{H}$  and that parallelizes the Lie algebra  $\mathcal{V}$ . At every point  $x \in \mathbb{M}$ , we can find a local frame of vector fields  $\{X_1, \dots, X_d, Z_1, \dots, Z_h\}$  such that on a neighborhood of  $x$ :

- (a)  $\{X_1, \dots, X_d\}$  is a  $g_{\mathcal{H}}$ -orthonormal basis of  $\mathcal{H}$ ;
- (b)  $\{Z_1, \dots, Z_h\}$  is a  $g_{\mathcal{V}}$ -orthonormal basis of the Lie algebra  $\mathcal{V}$ ;

Such a frame will be called a local adapted frame.

The sub-Laplacian on  $\mathbb{M}$  is the second-order differential operator which is given in a local adapted frame by

$$L = \sum_{i=1}^d \nabla_{X_i} \nabla_{X_i} - \nabla_{\nabla_{X_i} X_i}. \quad (2.1)$$

By declaring a one-form horizontal (resp. vertical) if it vanishes on the vertical bundle  $\mathcal{V}$  (resp. on the horizontal bundle  $\mathcal{H}$ ), the splitting of the tangent space

$$T_x \mathbb{M} = \mathcal{H}(x) \oplus \mathcal{V}(x)$$

gives a splitting of the cotangent space

$$T_x^* \mathbb{M} = \mathcal{H}^*(x) \oplus \mathcal{V}^*(x).$$

If  $\{X_1, \dots, X_d, Z_1, \dots, Z_h\}$  is a local adapted frame, the dual frame will be denoted  $\{\theta_1, \dots, \theta_d, \nu_1, \dots, \nu_h\}$  and referred to as a local adapted coframe. With a slight abuse of notations, for  $\varepsilon > 0$ , the metric on  $T_x^* \mathbb{M}$  that makes  $\{\theta_1, \dots, \theta_d, \frac{1}{\sqrt{\varepsilon}} \nu_1, \dots, \frac{1}{\sqrt{\varepsilon}} \nu_h\}$  orthonormal will still be denoted  $g_\varepsilon$  or  $\langle \cdot, \cdot \rangle_\varepsilon$ . This metric on the cotangent bundle can thus be written

$$g_\varepsilon = g_{\mathcal{H}^*} \oplus \varepsilon g_{\mathcal{V}^*}, \quad \varepsilon > 0, \quad (2.2)$$

where  $g_{\mathcal{H}^*}$  (resp.  $g_{\mathcal{V}^*}$ ) is the metric on  $\mathcal{H}^*$  (resp.  $\mathcal{V}^*$ ) that makes  $\{\theta_1, \dots, \theta_d\}$  (resp.  $\{\nu_1, \dots, \nu_h\}$ ) orthonormal. We use similar notations and conventions as before so that for every  $\eta$  in  $T_x^* \mathbb{M}$ ,

$$\|\eta\|_\varepsilon^2 = \|\eta\|_{\mathcal{H}^*}^2 + \varepsilon \|\eta\|_{\mathcal{V}^*}^2.$$

We now introduce some tensors that will play an important role in the sequel. We define  $\mathfrak{Ric}_{\mathcal{H}} : T_x^*\mathbb{M} \rightarrow T_x^*\mathbb{M}$  as the symmetric linear map on one forms such that for every smooth functions  $f, g$ ,

$$\langle \mathfrak{Ric}_{\mathcal{H}}(df), dg \rangle_{\mathcal{H}^*} = \mathbf{Ricci}(\nabla_{\mathcal{H}}f, \nabla_{\mathcal{H}}g),$$

where  $\mathbf{Ricci}$  is the Ricci curvature of the connection  $\nabla$  and  $\nabla_{\mathcal{H}}$  the horizontal gradient (projection of the gradient on the horizontal distribution  $\mathcal{H}$ ). Similarly, we will denote by  $\nabla_{\mathcal{V}}$  the vertical gradient, that is the projection of the gradient on the vertical bundle. In a local adapted frame  $\{X_1, \dots, X_d, Z_1, \dots, Z_{\mathfrak{h}}\}$ , we have thus

$$\begin{aligned}\nabla_{\mathcal{H}}f &= \sum_{i=1}^d (X_i f) X_i, \\ \nabla_{\mathcal{V}}f &= \sum_{m=1}^{\mathfrak{h}} (Z_m f) Z_m.\end{aligned}$$

and

$$\mathbf{Ricci}(\nabla_{\mathcal{H}}f, \nabla_{\mathcal{H}}g) = \sum_{n=1}^d g_{\mathcal{H}}(\mathbf{R}(\nabla_{\mathcal{H}}f, X_n)X_n, \nabla_{\mathcal{H}}g),$$

where  $\mathbf{R}$  is the Riemannian curvature tensor:  $\mathbf{R}(X_i, X_j)X_k = \nabla_{X_i}\nabla_{X_j}X_k - \nabla_{X_j}\nabla_{X_i}X_k - \nabla_{[X_i, X_j]}X_k$ .

For  $Z \in \mathcal{V}$ , we consider the unique skew-symmetric map  $J_Z$  defined on the horizontal bundle  $\mathcal{H}$  such that for every horizontal vector fields  $X$  and  $Y$ ,

$$\langle J_Z(X), Y \rangle_{\mathcal{H}} = \langle Z, T(X, Y) \rangle_{\mathcal{V}}. \quad (2.3)$$

We can then extend  $J_Z$  to the whole tangent space  $T_x\mathbb{M}$  by imposing that  $J_Z(V) = 0$  whenever  $V$  is a vertical vector field. If  $(Z_m)_{1 \leq m \leq \mathfrak{h}}$  is a  $g_{\mathcal{V}}$ -orthonormal basis of the Lie algebra  $\mathcal{V}$ , the operator  $\sum_{m=1}^{\mathfrak{h}} J_{Z_m}^* J_{Z_m} = -\sum_{m=1}^{\mathfrak{h}} J_{Z_m}^2 : T_x\mathbb{M} \rightarrow T_x\mathbb{M}$  does not depend on the choice of the basis and will concisely be denoted by  $-\mathbf{J}^2$ . We can note that in the case where  $\mathbb{M}$  is a Sasakian manifold,  $\mathbf{J}^2 = -\mathfrak{h}\mathbf{Id}_{\mathcal{H}}$ . Though originally defined on vector fields we will also consider  $-\mathbf{J}^2$  as the linear map  $T_x^*\mathbb{M} \rightarrow T_x^*\mathbb{M}$  defined by

$$\langle -\mathbf{J}^2(\theta_i), \theta_j \rangle_{\mathcal{H}^*} = \langle -\mathbf{J}^2(X_i), X_j \rangle_{\mathcal{H}}, \quad 1 \leq i, j \leq d$$

Then  $-\mathbf{J}^2$  is defined to be 0 on vertical one-forms.

If  $V$  is a horizontal vector field, then we consider an operator  $\mathfrak{T}_V^{\varepsilon}$  on smooth sections of the cotangent bundle given by

$$\mathfrak{T}_V^{\varepsilon}\eta = -\sum_{j=1}^d \eta(T(V, X_j))\theta_j + \frac{1}{2\varepsilon} \sum_{m=1}^{\mathfrak{h}} \eta(J_{Z_m}V)\nu_m$$

in a local frame. It is easily seen that  $\mathfrak{T}_V^\varepsilon$  is a skew-symmetric operator for the metric  $g_{2\varepsilon}$  that was previously defined on one-forms by (2.2).

If  $\eta$  is a one-form, we define the horizontal gradient in a local adapted frame of  $\eta$  as the  $(0, 2)$  tensor

$$\nabla_{\mathcal{H}}\eta = \sum_{i=1}^d \nabla_{X_i}\eta \otimes \theta_i.$$

Similarly, we will use the notation

$$\mathfrak{T}_{\mathcal{H}}^\varepsilon\eta = \sum_{i=1}^d \mathfrak{T}_V^\varepsilon\eta \otimes \theta_i.$$

We finally recall the following definition that was introduced in [4]:

**Definition 2.3** *The sub-Riemannian manifold  $\mathbb{M}$  is said to be of Yang-Mills type, if the horizontal divergence of the torsion vanishes that is for every horizontal vector field  $X$ , and every adapted local frame*

$$\sum_{\ell=1}^d (\nabla_{X_\ell} T)(X_\ell, X) = 0.$$

There are many interesting examples of Yang-Mills sub-Riemannian manifolds with transverse symmetries (see [4]). Sasakian and 3-Sasakian manifolds are examples of Yang-Mills sub-Riemannian manifolds. Though not identical, the Yang-Mills condition can be compared to the divergence free torsion condition that was considered in [18].

The following Bochner Weitzenböck formula was proved in [3] to which we refer for further details.

**Theorem 2.4 (Bochner-Weitzenböck formula [3])** *Assume that  $\mathbb{M}$  is a sub-Riemannian manifold with transverse symmetries of Yang-Mills type. For  $\varepsilon > 0$ , we consider the  $g_{2\varepsilon}$ -self-adjoint operator which is defined on one-forms by*

$$\square_\varepsilon = -(\nabla_{\mathcal{H}} - \mathfrak{T}_{\mathcal{H}}^\varepsilon)^*(\nabla_{\mathcal{H}} - \mathfrak{T}_{\mathcal{H}}^\varepsilon) - \frac{1}{2\varepsilon} \mathbf{J}^2 - \mathfrak{Ric}_{\mathcal{H}}.$$

*Then, for every smooth function  $f$  on  $\mathbb{M}$ ,*

$$d(Lf) = \square_\varepsilon(df),$$

*and for any smooth one-form  $\eta$ ,*

$$\frac{1}{2}L\|\eta\|_{2\varepsilon}^2 - \langle \square_\varepsilon\eta, \eta \rangle_{2\varepsilon} = \|\nabla_{\mathcal{H}}\eta - \mathfrak{T}_{\mathcal{H}}^\varepsilon\eta\|_{2\varepsilon}^2 + \left\langle \mathfrak{Ric}_{\mathcal{H}}(\eta) + \frac{1}{2\varepsilon} \mathbf{J}^2(\eta), \eta \right\rangle_{\mathcal{H}^*}.$$

In the previous statement  $(\nabla_{\mathcal{H}} - \mathfrak{T}_{\mathcal{H}}^\varepsilon)^*$  is understood as an adjoint for the  $g_{2\varepsilon}$ -metric and it is easily seen (see [3]) that in a local adapted frame, we have

$$-(\nabla_{\mathcal{H}} - \mathfrak{T}_{\mathcal{H}}^\varepsilon)^*(\nabla_{\mathcal{H}} - \mathfrak{T}_{\mathcal{H}}^\varepsilon) = \sum_{i=1}^d (\nabla_{X_i} - \mathfrak{T}_{X_i}^\varepsilon)^2 - (\nabla_{\nabla_{X_i} X_i} - \mathfrak{T}_{\nabla_{X_i} X_i}^\varepsilon),$$

and for any smooth one-form  $\eta$ ,

$$\|\nabla_{\mathcal{H}} \eta - \mathfrak{T}_{\mathcal{H}}^\varepsilon \eta\|_{2\varepsilon}^2 = \sum_{i=1}^d \|\nabla_{X_i} \eta - \mathfrak{T}_{X_i}^\varepsilon \eta\|_{2\varepsilon}^2.$$

### 3 Lichnerowicz estimate

From now on, we consider a compact Yang-Mills sub-Riemannian manifold  $\mathbb{M}$  with transverse symmetries and adopt the conventions and notations of the previous section. In particular  $L$  denotes the sub-Laplacian on  $\mathbb{M}$ . In this section, we prove the following result.

**Theorem 3.1** *Assume that for every smooth horizontal one-form  $\eta$ ,*

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta \rangle_{\mathcal{H}^*} \geq \rho_1 \|\eta\|_{\mathcal{H}^*}^2, \quad \langle -\mathbf{J}^2(\eta), \eta \rangle_{\mathcal{H}^*} \leq \kappa \|\eta\|_{\mathcal{H}^*}^2,$$

*and that for every  $Z \in \mathcal{V}$ ,*

$$\mathbf{Tr}(J_Z^* J_Z) \geq \rho_2 \|Z\|_{\mathcal{V}}^2,$$

*with  $\rho_1, \rho_2 > 0$  and  $\kappa \geq 0$ . Then the first eigenvalue  $\lambda_1$  of the sub-Laplacian  $-L$  satisfies*

$$\lambda_1 \geq \frac{\rho_1}{1 - \frac{1}{d} + \frac{3\kappa}{\rho_2}}.$$

Before we prove the result, we briefly discuss the argument that was used in [5] to quickly get, under the same assumptions, a lower bound on  $\lambda_1$  which is less sharp.

If  $f$  is a smooth function on  $\mathbb{M}$ , then we have from Theorem 2.4

$$\frac{1}{2} L \|df\|_{2\varepsilon}^2 - \langle d(Lf), df \rangle_{2\varepsilon} = \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^\varepsilon df\|_{2\varepsilon}^2 + \left\langle \mathfrak{Ric}_{\mathcal{H}}(df) + \frac{1}{2\varepsilon} \mathbf{J}^2(df), df \right\rangle_{\mathcal{H}^*}.$$

Integrating this equality over  $\mathbb{M}$  and using the assumptions

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta \rangle_{\mathcal{H}^*} \geq \rho_1 \|\eta\|_{\mathcal{H}^*}^2, \quad \langle -\mathbf{J}^2(\eta), \eta \rangle_{\mathcal{H}^*} \leq \kappa \|\eta\|_{\mathcal{H}^*}^2,$$

we deduce

$$- \int_{\mathbb{M}} \langle d(Lf), df \rangle_{2\varepsilon} \geq \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^\varepsilon df\|_{2\varepsilon}^2 + \left( \rho_1 - \frac{\kappa}{2\varepsilon} \right) \int_{\mathbb{M}} \|df\|_{\mathcal{H}^*}^2.$$

An integration by parts of left hand side of the inequality gives then

$$\int_{\mathbb{M}} (Lf)^2 - 2\varepsilon \int_{\mathbb{M}} \langle d(Lf), df \rangle_{\mathcal{V}^*} \geq \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{2\varepsilon}^2 + \left(\rho_1 - \frac{\kappa}{2\varepsilon}\right) \int_{\mathbb{M}} \|df\|_{\mathcal{H}^*}^2. \quad (3.4)$$

Now, a straightforward application of the Cauchy-Schwarz inequality yields the pointwise lower bound

$$\|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{\mathcal{H}^*}^2 \geq \frac{1}{d} (Lf)^2 + \frac{1}{4} \rho_2 \|df\|_{\mathcal{V}^*}^2. \quad (3.5)$$

Coming back to (3.4), we infer then

$$\frac{d-1}{d} \int_{\mathbb{M}} (Lf)^2 - 2\varepsilon \int_{\mathbb{M}} \langle d(Lf), df \rangle_{\mathcal{V}^*} \geq \left(\rho_1 - \frac{\kappa}{2\varepsilon}\right) \int_{\mathbb{M}} \|df\|_{\mathcal{H}^*}^2 + \frac{1}{4} \rho_2 \int_{\mathbb{M}} \|df\|_{\mathcal{V}^*}^2.$$

In particular, if  $Lf = -\lambda_1 f$ , then we obtain

$$\frac{d-1}{d} \lambda_1^2 \int_{\mathbb{M}} f^2 + 2\varepsilon \lambda_1 \int_{\mathbb{M}} \|df\|_{\mathcal{V}^*}^2 \geq \left(\rho_1 - \frac{\kappa}{2\varepsilon}\right) \lambda_1 \int_{\mathbb{M}} f^2 + \frac{1}{4} \rho_2 \int_{\mathbb{M}} \|df\|_{\mathcal{V}^*}^2.$$

Choosing  $\varepsilon$  such that  $2\varepsilon\lambda_1 = \frac{1}{4}\rho_2$  yields

$$\lambda_1 \geq \frac{\rho_1}{1 - \frac{1}{d} + \frac{4\kappa}{\rho_2}}.$$

This is not the optimal lower bound we are looking for. It is possible to improve this lower bound from (3.4) by first integrating by parts the term  $\int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{2\varepsilon}^2$  and, then using Cauchy-Schwarz inequality. The key lemma is the following:

**Lemma 3.2** For  $f \in C^\infty(\mathbb{M})$ ,

$$\begin{aligned} \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{2\varepsilon}^2 &= \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{\mathcal{H}^*}^2 + 2\varepsilon \int_{\mathbb{M}} \left\| \nabla_{\mathcal{H}} df - \frac{3}{2} \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df \right\|_{\mathcal{V}^*}^2 \\ &\quad + \frac{1}{2} \int_{\mathbb{M}} \mathbf{Tr}(J_{\nabla_{\mathcal{V}} f}^* J_{\nabla_{\mathcal{V}} f}) - \frac{5}{2} \varepsilon \int_{\mathbb{M}} \|\mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{\mathcal{V}^*}^2. \end{aligned}$$

*Proof.* Using the definition  $\mathfrak{T}_{\mathcal{H}}^{\varepsilon}$  together with the Yang-Mills assumption, we see that

$$\int_{\mathbb{M}} \langle \nabla_{\mathcal{H}} df, \mathfrak{T}_{\mathcal{H}}^{\varepsilon}(df) \rangle_{\mathcal{V}^*} = \frac{1}{4\varepsilon} \int_{\mathbb{M}} \mathbf{Tr}(J_{\nabla_{\mathcal{V}} f}^* J_{\nabla_{\mathcal{V}} f}). \quad (3.6)$$

As a consequence, we obtain

$$\begin{aligned} &\int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{2\varepsilon}^2 \\ &= \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{\mathcal{H}^*}^2 + 2\varepsilon \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{\mathcal{V}^*}^2 \\ &= \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{\mathcal{H}^*}^2 + 2\varepsilon \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df\|_{\mathcal{V}^*}^2 - 4\varepsilon \int_{\mathbb{M}} \langle \nabla_{\mathcal{H}} df, \mathfrak{T}_{\mathcal{H}}^{\varepsilon}(df) \rangle_{\mathcal{V}^*} + 2\varepsilon \int_{\mathbb{M}} \|\mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{\mathcal{V}^*}^2 \end{aligned} \quad (3.7)$$



By using (3.6), the trick is now to write

$$\begin{aligned}\int_{\mathbb{M}} \langle \nabla_{\mathcal{H}} df, \mathfrak{T}_{\mathcal{H}}^{\varepsilon}(df) \rangle_{\mathcal{V}^*} &= \frac{3}{2} \int_{\mathbb{M}} \langle \nabla_{\mathcal{H}} df, \mathfrak{T}_{\mathcal{H}}^{\varepsilon}(df) \rangle_{\mathcal{V}^*} - \frac{1}{2} \int_{\mathbb{M}} \langle \nabla_{\mathcal{H}} df, \mathfrak{T}_{\mathcal{H}}^{\varepsilon}(df) \rangle_{\mathcal{V}^*} \\ &= \frac{3}{2} \int_{\mathbb{M}} \langle \nabla_{\mathcal{H}} df, \mathfrak{T}_{\mathcal{H}}^{\varepsilon}(df) \rangle_{\mathcal{V}^*} - \frac{1}{8\varepsilon} \int_{\mathbb{M}} \mathbf{Tr}(J_{\nabla_{\mathcal{V}} f}^* J_{\nabla_{\mathcal{V}} f}).\end{aligned}$$

Coming back to (3.7) and completing the squares gives

$$\begin{aligned}\int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{2\varepsilon}^2 &= \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{\mathcal{H}^*}^2 + 2\varepsilon \int_{\mathbb{M}} \left\| \nabla_{\mathcal{H}} df - \frac{3}{2} \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df \right\|_{\mathcal{V}^*}^2 \\ &\quad + \frac{1}{2} \int_{\mathbb{M}} \mathbf{Tr}(J_{\nabla_{\mathcal{V}} f}^* J_{\nabla_{\mathcal{V}} f}) - \frac{5}{2}\varepsilon \int_{\mathbb{M}} \|\mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{\mathcal{V}^*}^2.\end{aligned}$$

□

We are now in position to complete the proof of Theorem 3.1.

*Proof.* Using the previous Lemma, Cauchy-Schwarz inequality and the assumptions

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta \rangle_{\mathcal{H}^*} \geq \rho_1 \|\eta\|_{\mathcal{H}^*}^2, \quad \langle -\mathbf{J}^2(\eta), \eta \rangle_{\mathcal{H}^*} \leq \kappa \|\eta\|_{\mathcal{H}^*}^2, \quad \mathbf{Tr}(J_Z^* J_Z) \geq \rho_2 \|Z\|_{\mathcal{V}}^2$$

we get the lower bound

$$\int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{2\varepsilon}^2 \geq \frac{1}{d} \int_{\mathbb{M}} (Lf)^2 + \frac{3}{4} \rho_2 \int_{\mathbb{M}} \|df\|_{\mathcal{V}^*}^2 - \frac{5}{8\varepsilon} \kappa \int_{\mathbb{M}} \|df\|_{\mathcal{H}^*}^2.$$

From (3.4), we know that

$$\int_{\mathbb{M}} (Lf)^2 - 2\varepsilon \int_{\mathbb{M}} \langle d(Lf), df \rangle_{\mathcal{V}^*} \geq \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{2\varepsilon}^2 + \left( \rho_1 - \frac{\kappa}{2\varepsilon} \right) \int_{\mathbb{M}} \|df\|_{\mathcal{H}^*}^2. \quad (3.8)$$

We thus deduce

$$\frac{d-1}{d} \int_{\mathbb{M}} (Lf)^2 - 2\varepsilon \int_{\mathbb{M}} \langle d(Lf), df \rangle_{\mathcal{V}^*} \geq \left( \rho_1 - \frac{9\kappa}{8\varepsilon} \right) \int_{\mathbb{M}} \|df\|_{\mathcal{H}^*}^2 + \frac{3}{4} \rho_2 \int_{\mathbb{M}} \|df\|_{\mathcal{V}^*}^2.$$

Now if  $f$  satisfies  $Lf = -\lambda_1 f$ , we get

$$\frac{d-1}{d} \lambda_1^2 \int_{\mathbb{M}} f^2 + 2\varepsilon \lambda_1 \int_{\mathbb{M}} \|df\|_{\mathcal{V}^*}^2 \geq \left( \rho_1 - \frac{9\kappa}{8\varepsilon} \right) \lambda_1 \int_{\mathbb{M}} f^2 + \frac{3}{4} \rho_2 \int_{\mathbb{M}} \|df\|_{\mathcal{V}^*}^2.$$

Choosing  $\varepsilon$  such that

$$2\varepsilon \lambda_1 = \frac{3}{4} \rho_2,$$

the desired lower bound on  $\lambda_1$  is obtained. □

## 4 The Obata sphere theorem on $H$ -type manifolds

In this section we prove the optimality of the lower bound for the first eigenvalue of the sub-Laplacian on a special class of Yang-Mills manifolds by obtaining a rigidity result in the spirit of the Obata sphere theorem.

We first introduce the following definition inspired from the notion of  $H$ -type groups that was introduced by Kaplan [20].

**Definition 4.1** *Let  $\mathbb{M}$  be a sub-Riemannian manifold with transverse symmetries of Yang-Mills type. We will say that  $\mathbb{M}$  is of  $H$ -type if for every  $Z \in \mathcal{V}$ ,  $\|Z\|_{\mathcal{V}} = 1$ , the map  $J_Z$  is orthogonal, that is,  $\langle J_Z(X), J_Z(Y) \rangle_{\mathcal{H}} = \langle X, Y \rangle_{\mathcal{H}}$  for  $X, Y \in \mathcal{H}(x)$ .*

Sasakian or 3-Sasakian manifolds are examples of  $H$ -type manifolds. If  $\mathbb{M}$  is a  $H$ -type sub-Riemannian manifold, it is immediate from the definition that for  $Z, Z' \in \mathcal{V}$ ,

$$J_Z J_{Z'} + J_{Z'} J_Z = -2\langle Z, Z' \rangle_{\mathcal{V}} \mathbf{Id}_{\mathcal{H}}.$$

In particular, we have

$$J_Z^2 = -\|Z\|_{\mathcal{V}}^2 \mathbf{Id}_{\mathcal{H}}.$$

In this section, we prove the following result:

**Theorem 4.2** *Let  $\mathbb{M}$  be a compact sub-Riemannian manifold of  $H$ -type. Assume that for every smooth horizontal one-form  $\eta$ ,*

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta \rangle_{\mathcal{H}^*} \geq \rho \|\eta\|_{\mathcal{H}^*}^2,$$

*with  $\rho > 0$ , then the first eigenvalue  $\lambda_1$  of the sub-Laplacian  $-L$  satisfies*

$$\lambda_1 \geq \frac{\rho d}{d-1+3\mathfrak{h}}.$$

*Moreover, if  $\lambda_1 = \frac{\rho d}{d-1+3\mathfrak{h}}$ , then  $\mathbb{M}$  is equivalent to a 1-Sasakian sphere  $\mathbb{S}^{2m+1}(r)$  or a 3-Sasakian sphere  $\mathbb{S}^{4m+3}(r)$  for some  $r > 0$  and  $m \geq 1$ .*

To put things in perspective, we pause a little and describe the sub-Riemannian geometry of the 1 and 3 Sasakian spheres (see for instance [6, 8] for more details) and precise what we mean by equivalent in the previous theorem.

- The sub-Riemannian geometry of the standard 1-Sasakian sphere  $\mathbb{S}^{2m+1}(1)$  is induced from the Riemannian structure of the complex projective space  $\mathbb{CP}^m$  by the Hopf fibration  $\mathbf{U}(1) \rightarrow \mathbb{S}^{2m+1} \rightarrow \mathbb{CP}^m$ . The sub-Laplacian  $L$  is then the lift of the Laplace-Beltrami operator on  $\mathbb{CP}^m$ . In that case,  $\lambda_1 = 2m$ .
- The sub-Riemannian geometry of the standard 3-Sasakian sphere  $\mathbb{S}^{4m+3}$  is induced from the Riemannian structure of the quaternionic projective space  $\mathbb{HP}^m$  by the quaternionic Hopf fibration  $\mathbf{SU}(2) \rightarrow \mathbb{S}^{4m+3} \rightarrow \mathbb{HP}^m$ . The sub-Laplacian  $L$  is then the lift of the Laplace-Beltrami operator on  $\mathbb{HP}^m$ . In that case,  $\lambda_1 = m$ .

In the previous theorem, we use the following notion of equivalence for sub-Riemannian manifolds with transverse symmetries: Two sub-Riemannian manifolds with transverse symmetries  $(\mathbb{M}_1, \mathcal{H}_1, \mathcal{V}_1)$  and  $(\mathbb{M}_2, \mathcal{H}_2, \mathcal{V}_2)$  are said to be equivalent if there exists a diffeomorphism  $\mathbb{M}_1 \rightarrow \mathbb{M}_2$  that induces an isometry between the horizontal distributions  $\mathcal{H}_1$  and  $\mathcal{H}_2$  and a Lie algebra isomorphism between  $\mathcal{V}_1$  and  $\mathcal{V}_2$ .

We now discuss the cases that were already known in the literature. As we pointed out Sasakian manifolds are of  $H$ -type. In that case  $\mathfrak{h} = 1$  and the lower bound becomes

$$\lambda_1 \geq \frac{\rho d}{d+2}.$$

This estimate was obtained by Greenleaf [12] (see also [2]). The estimate is optimal and the corresponding Obata's type rigidity result was obtained in [9] (see also [18] and [21]). The other case that was studied in the literature is the case of 3-Sasakian manifolds for which  $\mathfrak{h} = 3$ . The lower bound is then

$$\lambda_1 \geq \frac{\rho d}{d+8}.$$

This bound was proved in [15, 16] and the corresponding rigidity result was obtained in [17].

We now turn to the proof of Theorem 4.2. From now on, in the sequel,  $\mathbb{M}$  will be a compact sub-Riemannian manifold of  $H$ -type such that for every smooth horizontal one-form  $\eta$ ,

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta \rangle_{\mathcal{H}^*} \geq \rho \|\eta\|_{\mathcal{H}^*}^2,$$

with  $\rho > 0$ . Since  $\mathbb{M}$  is of  $H$ -type, we have

$$\langle -\mathbf{J}^2(\eta), \eta \rangle_{\mathcal{H}^*} = \mathfrak{h} \|\eta\|_{\mathcal{H}^*}^2,$$

and for every  $Z \in \mathcal{V}$ ,

$$\mathbf{Tr}(J_Z^* J_Z) = d \|Z\|_{\mathcal{V}}^2.$$

From Theorem 3.1, we get therefore the lower bound

$$\lambda_1 \geq \frac{\rho d}{d-1+3\mathfrak{h}}.$$

The key lemma in our rigidity result is the following result:

**Lemma 4.3** *Let  $f \in C^\infty(\mathbb{M})$  such that  $Lf = -\lambda_1 f$  with  $\lambda_1 = \frac{\rho d}{d-1+3\mathfrak{h}}$ . Then  $f$  satisfies*

$$\nabla^2 f(X, Y) = -\frac{\lambda_1}{d} f \langle X, Y \rangle_{\mathcal{H}} - \frac{1}{2} T(X, Y) f, \quad \forall X, Y \in \mathcal{H}. \quad (4.9)$$

and

$$\nabla^2 f(X, Z) = \frac{2\lambda_1}{\rho^2} J_Z(X) f, \quad \forall X \in \mathcal{H}, Z \in \mathcal{V}. \quad (4.10)$$

*Proof.* From (3.8) we have

$$\int_{\mathbb{M}} (Lf)^2 - 2\varepsilon \int_{\mathbb{M}} \langle d(Lf), df \rangle_{\mathcal{V}^*} \geq \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{2\varepsilon}^2 + \left(\rho_1 - \frac{\kappa}{2\varepsilon}\right) \int_{\mathbb{M}} \|df\|_{\mathcal{H}^*}^2,$$

and thus, since  $Lf = -\lambda_1 f$ ,

$$\lambda_1^2 \int_{\mathbb{M}} f^2 + 2\lambda_1 \varepsilon \int_{\mathbb{M}} \|df\|_{\mathcal{V}^*}^2 \geq \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{2\varepsilon}^2 + \lambda_1 \left(\rho_1 - \frac{\kappa}{2\varepsilon}\right) \int_{\mathbb{M}} f^2. \quad (4.11)$$

On the other hand, from Lemma 3.2, we have

$$\begin{aligned} \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{2\varepsilon}^2 &\geq \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{\mathcal{H}^*}^2 + 2\varepsilon \int_{\mathbb{M}} \left\| \nabla_{\mathcal{H}} df - \frac{3}{2} \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df \right\|_{\mathcal{V}^*}^2 \\ &\quad + \frac{\rho_2}{2} \int_{\mathbb{M}} \|df\|_{\mathcal{V}^*}^2 - \frac{5}{8\varepsilon} \int_{\mathbb{M}} \|df\|_{\mathcal{H}^*}^2. \end{aligned}$$

It is readily checked that

$$\|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{\mathcal{H}^*}^2 = \|\nabla_{\mathcal{H}}^{2,\#} f\|^2 + \frac{1}{4} \text{Tr}(J_{\nabla_{\mathcal{V}} f}^* J_{\nabla_{\mathcal{V}} f}),$$

where  $\nabla_{\mathcal{H}}^{2,\#} f$  denotes the symmetrization of the horizontal Hessian of  $f$ . Thus we have

$$\int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{2\varepsilon}^2 \geq \int_{\mathbb{M}} \|\nabla_{\mathcal{H}}^{2,\#} f\|^2 + 2\varepsilon \int_{\mathbb{M}} \left\| \nabla_{\mathcal{H}} df - \frac{3}{2} \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df \right\|_{\mathcal{V}^*}^2 + \frac{3\rho_2}{4} \int_{\mathbb{M}} \|df\|_{\mathcal{V}^*}^2 - \frac{5}{8\varepsilon} \lambda_1 \int_{\mathbb{M}} f^2.$$

Choosing  $\varepsilon$  such that  $2\lambda_1 \varepsilon = \frac{3\rho_2}{4}$  and using the last inequality in (4.11) gives eventually

$$\left( \lambda_1^2 - \lambda_1 \left( \rho_1 - \frac{9\kappa}{8\varepsilon} \right) \right) \int_{\mathbb{M}} f^2 \geq \int_{\mathbb{M}} \|\nabla_{\mathcal{H}}^{2,\#} f\|^2 + 2\varepsilon \int_{\mathbb{M}} \left\| \nabla_{\mathcal{H}} df - \frac{3}{2} \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df \right\|_{\mathcal{V}^*}^2.$$

From Cauchy-Schwarz inequality, given the value of  $\lambda_1$ , we always have

$$\left( \lambda_1^2 - \lambda_1 \left( \rho_1 - \frac{9\kappa}{8\varepsilon} \right) \right) \int_{\mathbb{M}} f^2 \leq \int_{\mathbb{M}} \|\nabla_{\mathcal{H}}^{2,\#} f\|^2$$

This means that, necessarily

$$\left\| \nabla_{\mathcal{H}} df - \frac{3}{2} \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df \right\|_{\mathcal{V}^*} = 0,$$

and moreover that  $\nabla_{\mathcal{H}}^{2,\#} f$  is a multiple of  $g_{\mathcal{H}}$ . This immediately implies (4.9) and (4.10).  $\square$

We are now in position to prove Theorem 4.2.

*Proof.* Let  $f \in C^\infty(\mathbb{M})$  such that  $Lf = -\lambda_1 f$  with  $\lambda_1 = \frac{\rho d}{d-1+3\mathfrak{h}}$ . From the previous lemma, we have

$$\nabla^2 f(X, Y) = -\frac{\lambda_1}{d} f \langle X, Y \rangle_{\mathcal{H}} - \frac{1}{2} T(X, Y) f, \quad \forall X, Y \in \mathcal{H}.$$

and

$$\nabla^2 f(X, Z) = \frac{2\lambda_1}{d} J_Z(X)f, \quad \forall X \in \mathcal{H}, Z \in \mathcal{V}.$$

The trick is now that, since  $\mathbb{M}$  has transverse symmetries,  $-L$  commutes with any  $Z \in \mathcal{V}$  (see [4]), and thus  $Zf$  is also an eigenfunction for the same eigenvalue  $\lambda_1$ . In particular  $Zf$  also satisfies the equation (4.10). This gives for a horizontal vector field  $X$  and  $Z \in \mathcal{V}$ ,

$$\nabla^3 f(X, Z, Z) = \frac{4\lambda_1^2}{d^2} J_Z^2(X)f.$$

From the  $H$ -type assumption, we deduce

$$\nabla^3 f(X, Z, Z) = -\frac{4\lambda_1^2}{d^2} \|Z\|_{\mathcal{V}}^2 Xf.$$

Taking the trace and using the fact that both  $f$  and  $Zf$  are eigenfunctions of  $-L$  with the same eigenvalue, we deduce that for any  $Z \in \mathcal{V}$ ,

$$Z^2 f = -\frac{4\lambda_1^2}{d^2} \|Z\|_{\mathcal{V}}^2 f.$$

By polarization, it also implies that for every  $Z, Z' \in \mathcal{V}$ ,

$$\frac{1}{2}(ZZ' + Z'Z)f = -\frac{4\lambda_1^2}{d^2} \langle Z, Z' \rangle_{\mathcal{V}} f. \quad (4.12)$$

Since  $\mathbb{M}$  is compact, we easily see that  $\mathcal{V}$  is a Lie algebra of compact type. We therefore can choose  $g_{\mathcal{V}}$  to be a bi-invariant metric. We consider then the Riemannian metric on  $\mathbb{M}$ ,

$$g_{2\varepsilon} = g_{\mathcal{H}} \oplus \frac{1}{2\varepsilon} g_{\mathcal{V}},$$

where  $\varepsilon = \frac{2\lambda_1}{d}$ . By denoting  $\tilde{\nabla}$  the Levi-Civita connection associated to  $g_{2\varepsilon}$ , it is then an easy exercise to check that the previous relations imply then that for every smooth vector fields  $X, Y$

$$\tilde{\nabla}^2 f(X, Y) = -\frac{\lambda_1}{d} f g_{2\varepsilon}(X, Y).$$

As a consequence of Obata's theorem [23], we deduce that  $(\mathbb{M}, g_{2\varepsilon})$  is isometric to a sphere. Also by the very same Obata's theorem, the relations (4.12) imply that the Lie group  $\mathbb{G}$  generated by  $\mathcal{V}$  is a sphere itself. This implies that this group is either  $\mathbf{U}(1)$  or  $\mathbf{SU}(2)$ . Moreover, by the very definition of sub-Riemannian manifolds with transverse symmetries,  $\mathbb{G}$  is seen to act properly on  $\mathbb{M}$ . We deduce that there is a Riemannian submersion with totally geodesic fibers

$$\mathbb{G} \rightarrow \mathbb{M} \rightarrow \mathbb{M}/\mathbb{G}.$$

The classification of Riemannian submersions with totally geodesic fibers of the sphere that was done in Escobales [11] completes our proof.  $\square$

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